THE CHROMATIC FRACTURE SQUARE: DISASSEMBLY AND REASSEMBLY

DORON GROSSMAN-NAPLES

Abstract. We discuss the chromatic fracture square and the basics of local chromatic homotopy theory.

Contents

The main source for this talk is Barthel and Beaudry's survey [2]. Other sources are cited as appropriate.

1. Chromatic Convergence

In this section, all spectra are p-local unless stated otherwise.

Recall that for any spectrum X, we can define its chromatic tower

$$\begin{array}{ccccccccc} & & M_nX & & & & M_2X & & M_1X & & M_0X \simeq H\mathbb{Q}\wedge X \\ & & \downarrow & & & & \downarrow & & \downarrow & & \| \\ \cdots & \longrightarrow & L_nX & \longrightarrow & \cdots & \longrightarrow & L_2X & \longrightarrow & L_1X & \longrightarrow & L_0X \simeq H\mathbb{Q}\wedge X, \end{array} \tag{1.1}$$

where we call M_nX the nth monochromatic layer of X and define it as the fiber of the evident map. This filtration is useful because, in many cases, it recovers the original spectrum.

Theorem 1.2 (Hopkins-Ravenel)**.** *If X is a finite spectrum, then the natural map $X \to \lim_n L_n(X)$ is an equivalence.*

Of course, this is not true for all spectra; for instance, the chromatic tower of $H\mathbb{F}_p$ is identically zero. But finite spectra aren't the only ones with this property.

Definition 1.3. The *chromatic completion* of a spectrum X is $\lim_n L_nX$. If $X \to \lim_n L_nX$ is an equivalence, we say X is *chromatically complete*.

Date: April 26th, 2023.

The term "complete" is in analogy to completion of a module at a prime, which has a formally similar definition. The theorem above states that finite spectra are chromatically complete. A larger class of chromatically complete spectra is described by Barthel in [1].

Theorem 1.4 (Barthel)**.** *If X is a connective spectrum with finite projective dimension over BP, X is chromatically complete.*

A related notion is that of a *harmonic* spectrum[1], defined and studied by Ravenel in [4].

Definition 1.5. A spectrum is *harmonic* if it is local with respect to $\bigvee_{n=0}^{\infty} K(n)$; we denote this localization functor by L_∞. A spectrum whose harmonic localization is trivial is called *dissonant*.

Some examples:

1. Finite spectra are harmonic ([4]).
2. Suspension spectra are harmonic ([3]). This implies that any simply-connected dissonant space is weakly contractible.
3. Chromatically complete spectra are harmonic, since they are a limit of harmonic spectra (their L_n-localizations).
4. Torsion spectra with $\pi_n = 0$ for $n \gg 0$ are always dissonant ([4]). In particular, this includes $H\mathbb{F}_p$.

One might reasonably ask if harmonic localization and chromatic completion are the same; in fact, Ravenel asked this very question. Barthel showed ([1]) that they are not equivalent, but they are related.

Theorem 1.6 (Barthel)**.**

(1) The spectrum $L_\infty \bigvee_{n=0}^{\infty} \Sigma^{n+1} C_n BP$ is harmonic but not chromatically complete, so harmonic localization is not equivalent to chromatic completion. (Here C_n is the acyclicization functor for L_n.)

(2) Harmonic localization is the idempotent approximation to chromatic completion, i.e. the terminal object in the category of idempotent monads mapping to chromatic completion.

Corollary 1.7. *Chromatic completion is not idempotent; that is, the chromatic completion of a spectrum is not always chromatically complete.*

(This is the same as the situation with the completion of modules in commutative algebra.)

2. The Fracture Square

Suppose we have a chromatically complete p-local spectrum X. We would like to assemble X from its monochromatic layers. While these spectra have some interesting properties ($M_n X$ is always a filtered colimit of periodic spectra with period a multiple of $2(p^n - 1)$, for instance), it is more convenient to work with the $K(n)$-localizations.

Consider the chromatic filtration

$$\mathrm{Sp} = \ker(0) \supset \ker(L_0) \supset \ker(L_1) \supset \cdots \supset \ker(\mathrm{id}) = (0)$$

[1] This terminology is quite unfortunate, since it conflicts very badly with Fourier analysis. As a result, it is rather difficult to find information about harmonic spectra using keyword searches.

and the associated ascending filtration

$$(0) = \mathrm{Im}(0) \subset \mathrm{Sp}_0 \subset \mathrm{Sp}_1 \subset \mathrm{Sp}_2 \subset \cdots \subset \mathrm{Im}(\mathrm{id}) = \mathrm{Sp}.$$

We can use these dual filtrations to realize the subquotients in two ways: as localizing subcategories (the monochromatic layers) or colocalizing subcategories ($K(n)$-localization). Thus we have a symmetric monoidal equivalence between $\mathrm{Sp}_{K(n)}$ and the image of M_n. (This is a particular case of a general duality called "local duality".) It follows that the $K(n)$-localization of X contains the same information as its nth chromatic layer. This is exhibited by the *chromatic fracture square.*

$$(2.1)\qquad \begin{array}{ccc} L_n X & \longrightarrow & L_{K(n)} X \\ \downarrow & & \downarrow \\ L_{n-1} X & \xrightarrow[\iota_n(X)]{} & L_{n-1} L_{K(n)} X \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{Sp}_n & \xrightarrow{L_{K(n)}} & \mathrm{Sp}_{K(n)} \\ {\scriptstyle X \mapsto \iota_n(X)}\downarrow & & \downarrow{\scriptstyle L_{n-1}} \\ \mathrm{Fun}(\Delta^1, \mathrm{Sp}_{n-1}) & \xrightarrow[\text{target}]{} & \mathrm{Sp}_{n-1} \end{array}$$

This is a topological manifestation of the stratification of $\mathcal{M}_{fg}^{\heartsuit}$, telling us how we deform from the $(n-1)$th layer to the nth. The process works both for individual spectra (which is computationally useful) and for categories of spectra (which is philosophically important). It tells us, up to a tower of extension problems, how to reassemble a p-local spectrum X from its $K(n)$-localizations.

The *chromatic splitting conjecture* predicts that the lower map of the left square splits (admits a section) when X is a finite p-complete spectrum. This would be very nice, since it would solve the extension problem for us. The conjecture is known to hold at heights ≤ 2, and is open above that. Personally, I find it a bit too good to be true, but that's just a gut feeling.

The chromatic fracture square is analogous to the arithmetic fracture square or "Hasse square", a classical local-to-global principle from number theory:

$$(2.2)\qquad \begin{array}{ccc} \mathbb{Z} & \longrightarrow & \prod_p \mathbb{Z}_p \\ \downarrow & & \downarrow \\ \mathbb{Q} & \longrightarrow & \mathbb{Q} \otimes \prod_p \mathbb{Z}_p \end{array} \qquad\qquad \begin{array}{ccc} M & \longrightarrow & \prod_p M_p^\wedge \\ \downarrow & & \downarrow \\ \mathbb{Q} \otimes M & \longrightarrow & \mathbb{Q} \otimes \prod_p M_p^\wedge . \end{array}$$

This fracture square also has a role to play in stable homotopy theory. By replacing the chain complex M with a finite spectrum X, we obtain a pullback square

$$\begin{array}{ccc} X & \longrightarrow & \prod_p X_p \\ \downarrow & & \downarrow \\ H\mathbb{Q} \wedge X & \longrightarrow & H\mathbb{Q} \wedge \prod_p X_p , \end{array}$$

which we can use to reassemble X from its p-complete parts.

In summary, our reassembly process is as follows.

(1) Understand the $K(n)$-local piece of X for each n.
(2) Solve the extension problem in the chromatic fracture square.

(3) Use chromatic convergence to recover $X_{(p)}$.
(4) Complete $X_{(p)}$ to X_p and use the arithmetic fracture square to recover X.

3. $K(n)$-Local Class Field Theory

The local story is most naturally described as an analogue of class field theory. Classically, class field theory studies number fields by looking at extensions of the local fields $\mathbb{Q}_p$. Thus, we start by describing Rognes's Galois theory for E_∞-rings ([5]), which I will henceforth just call "rings".

Definition 3.1. A map of rings $A \to B$ is called a *Galois extension with Galois group G* if G acts on B in CAlg_A in such a way that the canonical maps $A \to B^{hG}$ and $B \wedge B \to \prod_G B$ are both equivalences. A ring with no connected[2] Galois extensions for any nontrivial finite group G is called *separably closed.*

Minkowski's theorem in number theory states that every number field is ramified at at least one prime. It follows fairly quickly from this that $\mathbb{Z}$ is separably closed (see section 10 of Rognes). A corollary is the following:

Theorem 3.2. *The sphere spectrum $\mathbb{S}$ is separably closed.*

Proof. Suppose B is a finite Galois extension of $\mathbb{S}$. Then B is a dualizable $\mathbb{S}$-module, i.e. a finite spectrum, so its homology groups are finitely generated in each degree and zero in all but finitely many degrees. I claim that $H_n B = 0$ for $n \neq 0$. To show this, let n be minimal such that $\pi_n(B) \neq 0$, and suppose $n < 0$. Since $\mathbb{S} \to B$ is Galois, we have $H_n(B) \otimes H_n(B) \cong H_{2n}(B \wedge B) \cong \prod_G H_{2n}(B) = 0$. Thus by contradiction, $n = 0$. The $n > 0$ case is similar.

Write $T = H_0(B)$. Now the Hurewicz theorem implies that B is connective with $\pi_0(B) \cong H_0(B) = T$, and combining this with the Künneth formula shows that $T \otimes T \cong \prod_G T$ and $\mathrm{Tor}_1(T, T) = 0$. Thus T is a free abelian group of rank $|G|$. We can take the pushout of $\mathbb{S} \to B$ along the Hurewicz map Hurewicz map $\mathbb{S} \to H\mathbb{Z}$ to get a map $H\mathbb{Z} \to HT$. The $H\mathbb{Z}$-algebra HT is faithful since $\mathbb{Z} \to T$ is faithfully flat and dualizable since it is the pushout of a dualizable algebra, so a theorem of Rognes implies that it is a Galois extension. This implies that $\mathbb{Z} \to T$ is a Galois extension; but this means that either $T = \mathbb{Z}$, in which case G is trivial, or T is not connected, in which case B is not connected. Either way, we have a contradiction. □

This is a blow for our attempts to do algebra over the sphere spectrum (which is, after all, what stable homotopy theory is all about). Fortunately, this theorem becomes false after localization. This is where Morava E-theory enters the picture.

Recall that for any formal group Γ, say of height n over a perfect field k of characteristic p, its universal deformation is Landweber exact and yields a height n complex-oriented ring spectrum with a canonical E_∞ structure. This ring, E_Γ, is called the *Lubin-Tate spectrum* associated to Γ, and has an action by the associated *Morava stabilizer group*, a profinite group defined as $\mathbb{G}_\Gamma = \mathrm{Aut}(\Gamma) \rtimes \mathrm{Gal}(k/\mathbb{F}_p)$. In the case that Γ is the so-called "Honda formal group" of height n, Γ_n, we call this spectrum E_n ("height n Morava E-theory) and denote its Morava stabilizer group by $\mathbb{G}_n$.

[2]Connected here is in the sense of algebraic geometry; that is, B is connected if $\mathrm{Spec}\pi_0(B)$ is connected.

We need one more definition before we can state the main theorem. Since the Morava stabilizer group is profinite, we will need to describe an appropriate notion of "filtered colimit of Galois extensions".

Definition 3.3. Let $A \to B$ be a ring map, and let G be a profinite group acting on B in CAlg_A. Suppose that, for some cofiltered system of surjections (G_α) exhibiting G as a profinite group, the associated filtered system of fixed-point spectra B^{hG_α} consists entirely of Galois extensions with Galois groups given by the appropriate kernels. If B is the filtered colimit of these Galois extensions, we say that $A \to B$ is a *pro-Galois extension* with (pro-)Galois group G.

Theorem 3.4. *The unit map $L_{K(n)}\mathbb{S} \to E_\Gamma$ is a pro-Galois extension with Galois group $\mathbb{G}_\Gamma$. In particular, $L_{K(n)}\mathbb{S} \to E_n$ is a pro-Galois extension with Galois group $\mathbb{G}_n$.*

Aside from the incredible abstract beauty of this result, it is also useful for computations.

Theorem 3.5. *This Galois extension induces a fixed-point (or "descent") spectral sequence $E_2^{s,t} \cong H^s(\mathbb{G}_\Gamma, \pi_t(E_\Gamma)) \Longrightarrow \pi_{t-s}L_{K(n)}\mathbb{S}$. (Here we are taking the continuous group cohomology.) This spectral sequence coincides with the $K(n)$-local E_Γ-based Adams-Novikov spectral sequence. When $\Gamma = \Gamma_n$, this spectral sequence collapses with horizontal vanishing on a finite page.*

This is the starting point for doing computations with the ANSS in chromatic homotopy theory, and it works because the Galois theory gives us a nice expression for the E_2 page. (This expression applies not only for the sphere, but for any space X whose Morava module $\pi_* L_{K(n)}(E_n \wedge X)$ is sufficiently nice.) One powerful method for understanding this spectral sequence involves studying the subextensions associated to finite-index subgroups of $\mathbb{G}_n$. This is the theory of "finite resolutions", which we will discuss in our last meeting of the semester. Another approach at height 2 is to replace the Honda formal group with the formal group associated to a supersingular curve; this is one of the motivations for studying tmf.

Unfortunately, I don't have time to get into the details of the computations and homological algebra. I'll conclude instead by noting that this result, like the fracture square, can be categorified. The $\mathbb{G}_n$ action on E_n actually lifts to an action on Mod_{E_n}, yielding an equivalence of categories $\mathrm{Sp}_{K(n)} \simeq \mathrm{Mod}_{E_n}^{h\mathbb{G}_n}$. This suggests that the Lubin-Tate story plays a fundamental role in the structure of the stable homotopy category.

References

[1] Tobias Barthel. "Chromatic Completion". In: *Proceedings of the American Mathematical Society* 144.5 (2016), pp. 2263–2274. DOI: 10.48550/arXiv.1411.1711. arXiv: 1411.1711 [math]. URL: http://arxiv.org/abs/1411.1711.

[2] Tobias Barthel and Agnes Beaudry. "Chromatic Structures in Stable Homotopy Theory". In: *Handbook of Homotopy Theory*. Chapman and Hall/CRC, 2020, pp. 163–220.

[3] Michael J. Hopkins and Douglas C. Ravenel. "Suspension Spectra Are Harmonic". In: *Bol. Soc. Mat. Mexicana (2)* 37.1-2 (1992), pp. 271–279.

[4] Douglas C. Ravenel. “Localization with Respect to Certain Periodic Homology Theories”. In: *American Journal of Mathematics* 106.2 (Apr. 1984), p. 351. ISSN: 00029327. DOI: 10.2307/2374308. JSTOR: 2374308. URL: https://www.jstor.org/stable/2374308?origin=crossref (visited on 04/24/2023).

[5] John Rognes. *Galois Extensions of Structured Ring Spectra*. Dec. 6, 2005. DOI: 10.48550/arXiv.math/0502183. arXiv: math/0502183. URL: http://arxiv.org/abs/math/0502183 (visited on 08/12/2022). preprint.