TOWARDS A CHROMATIC LANGLANDS PROGRAM

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Abstract. Originally developed as a tool for computing the stable homotopy groups of spheres in topology, chromatic homotopy theory has proven to be highly interdisciplinary, possessing seemingly fundamental connections to number theory and the mathematics of high-energy physics. On the arithmetic side, it can be interpreted as a Brave New class field theory, with geometric models like tmf acting as a spectral version of arithmetic geometry. On the physical side, these complex-oriented cohomology theories act as receivers of "index maps" central to both concrete computations and geometry in quantum field theory. In this talk, I will propose a chromatic Langlands program that unifies notions of ramification, globalization, and equivariance appearing in these three fields. The approach taken involves modular and global equivariance for topological automorphic forms, cyclotomic trace as a description of geometric and algebraic ramification, and chromatic redshift; and, as suggested by the name, should be thought of as a spectral version of the Langlands program. Broadly speaking, the program aims to give a unified description of transchromatic geometry, and is expected to produce new computational tools in stable homotopy theory. Applications outside of topology include a rigorous interpretation of Witten's equivariant index theory and arithmetic geometry over the field with one element.

Contents

1. Arithmetic Globalization

In algebra, we study $\mathbb{Z}$ using the category of finitely-generated abelian groups. In stable homotopy theory, we study the sphere spectrum $\mathbb{S}$ using the category of finite spectra, Sp^ω. Since $\mathrm{Spec}(\mathbb{Z})$ is the Balmer spectrum of $\mathbb{Z}\,\mathrm{Mod}^\omega$, we think of $\mathrm{Spec}(\mathbb{S})$ as being the Balmer spectrum of Sp^ω (i.e. the moduli stack of formal groups), which looks like this:

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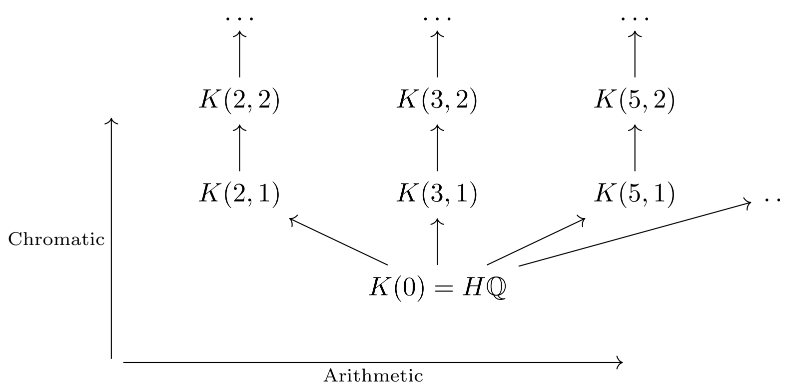

This stack has a vertical "chromatic direction" labeled by height $\in \mathbb{N}$ and a horizontal "arithmetic direction" labeled by prime in $\operatorname{Spec}\mathbb{Z}$. Traditionally, we would localize at some arithmetic prime p and get a category filtered by height. Then we can identify $K(n)$ as the "residue class field" at the point of height n and E_n as the "universal deformation about the height n stratum", i.e. the ring whose spec is the infinitesimal neighborhood of the height n point. We can use these to construct chromatic fracture squares which build up to the entire p-local category. We have thus "globalized" in the chromatic direction, yielding $MU_{(p)}$[1].

What if we instead globalize in the arithmetic direction? In his thesis ([12]), Mazel-Gee pointed out that this is precisely the procedure that gives us cohomology theories like tmf. More generally, by work of Behrens-Lawson ([5]), we should have a spectrum I call TAF_n, "topological automorphic forms", which acts as an arithmetically global version of E_n. At height 1, this is K-theory. At height 2, it is tmf. We see, moreover, that these are related; for instance, we have a canonical map $tmf \to KO[[q]]$, which is induced by the the Tate elliptic curve ([9]). After 2-localization, we can give a (connective) version of this that also involves ku:

$$\begin{array}{ccc} tmf_{(2)} & \longrightarrow & ko_{(2)} \\ \big\downarrow & & \big\downarrow \\ tmf_1(3)_{(2)} & \longrightarrow & ku_{(2)} \end{array} \quad ([11]).$$

These are all induced by maps of moduli stacks. We have maps $B\mathbb{G}_m \to B\mathbb{G}_{\mathrm{Tate}} \to \widehat{\mathcal{M}}_{\mathrm{ell}} \to \mathcal{M}_{\mathrm{fg}}$ which induce maps on the associated E_∞-rings by flatness and Goerss-Hopkins obstruction theory.

Notice that it is not only the global sections that are relevant here, but also sections of higher covers; for instance, KU is the sections of the double cover of

[1]More precisely, all of these are presentations of spectral stacks, so we really need to take their Hopf algebroids. This is why there are many different Morava E-theories coming from different formal groups: the universal deformation of any height n formal group will induce a Lubin-Tate spectrum which is a Galois extension of the $K(n)$-local sphere. The "standard" Morava E-theory is the version coming from the so-called Honda formal group, which is just a choice made for the sake of convenience.

$B\mathbb{G}_m$, and $Tmf_1(3)$ is the sections of the cover of $\widehat{\mathcal{M}}_{\text{ell}}$ associated to the congruence subgroup $\Gamma_1(3) \subset SL_2(\mathbb{Z})$. This is the first appearance of *level structures* in chromatic homotopy. In the theory of modular forms, a level structure is a choice of subgroup Γ of the modular group. Then modular forms with level Γ structure are not sections (of the relevant sheaf) over $\mathcal{M}_{\text{ell}}$, but rather over the étale cover associated to that subgroup.

The $K(n)$-localization of TAF_n is a product of homotopy fixed-point spectra of Lubin-Tate spectra associated to the universal deformations of certain arithmeto-geometrically described formal groups. The following theorem, which is Corollary 14.4.8 in [5], is a general version.

Theorem 1.1. *We have an isomorphism*

$$L_{K(n)}TAF_{GU}(K) \simeq \left(\prod_{x \in Sh^{[n]}(K)(\bar{\mathbb{F}_p})} E_n^{h\,\mathrm{Aut}(x)} \right)^{h\,\mathrm{Gal}(\bar{\mathbb{F}_p}/\mathbb{F}_p)}.$$

Here $TAF_{GU}(K)$ is a specific TAF spectrum whose construction I won't go into, and $SH^{[n]}(K)(\bar{\mathbb{F}_p})$ is the height n stratum of a certain Shimura variety.

For K-theory, this looks like

$$L_{K(1)}KO \simeq E_1^{hC_2 \rtimes \mathrm{Gal}(\bar{\mathbb{F}_p}/\mathbb{F}_p)}.$$

For tmf this looks like

$$L_{K(2)}TMF \simeq \left(\prod_{S \text{ supersingular elliptic curve}} E_n^{h\,\mathrm{Aut(S)}} \right)^{h\,\mathrm{Gal}(\bar{\mathbb{F}_p}/\mathbb{F}_p)}.$$

The reason this works is that a formal group Γ of height n has Landweber-exact universal deformation, which gives rise to a Lubin-Tate spectrum E_Γ of height n (which admits a canonical E_∞ structure by Goerss-Hopkins obstruction theory). This spectrum will be a Galois extension of the $K(n)$-local sphere, with Galois group given by the associated Morava stabilizer group $\mathbb{G}_\Gamma = \mathrm{Aut}(\Gamma) \rtimes \mathrm{Gal}(\bar{\mathbb{F}_p}/\mathbb{F}_p)$.

Thus we can think of TAF as something like an arithmetically global Galois extension of the height n sphere spectrum. This is surprising, since Rognes proved ([13]) that the global sphere spectrum is separably closed. Rognes's theorem also indicates that the expression "something like" is doing a lot of work here (as does the undefined term "height n sphere spectrum"). To interpret this, we can turn to number theory. The role of the spectrum E_n as a Galois extension of $L_{K(n)}\mathbb{S}$ is analogous to the role played by extensions of $\mathbb{Q}_p$ in local class field theory. In this metaphor, we should think of $\mathbb{S}$ as analogous to the integers, with complicated extensions that are most easily studied locally. If we want to study this theory globally, we can do so via the theory of modular Galois representations.

The following classical theorem is due to Eichler-Shimura for $k = 2$, Deligne for $k > 2$, and Deligne-Serre for $k = 1$.

Theorem 1.2. *Let $f \in S_k(\Gamma_0(N), \chi)$ be a newform*[2] *and ℓ prime. Write $f = \sum a(n)q^n$, and let K_λ be a finite extension of $\mathbb{Q}_\ell$ which contains all the $a(n)$s and the image of χ. Write $\mathcal{O}_\lambda$ for its ring of integers. Then there is a continuous*

[2] A modular form which vanishes at ∞ and spans a one-dimensional subrepresentation of the Hecke algebra.

irreducible representation $\rho_{f,\lambda} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathcal{O}_\lambda)$ *such that for any prime* $p \nmid N\ell$*,*

1. *This representation is unramified at* p*;*
2. *The trace of the* p*-Frobenius element is* $a(p)$*; and*
3. *The determinant of the* p*-Frobenius element is* $\chi(p)p^{k-1}$*.*

Generalizing this to arbitrary "heights" gives us the Langlands program: a fundamental correspondence between automorphic forms (more generally automorphic representations) and representations of Galois groups. The first key element of Chromatic Langlands is the fact that this is exactly what we have found in the theory of topological automorphic forms. We have automorphic representations (TAF with level structure) which give rise, after localization, to extensions of the local base (E_n) and a representation on some product of these extensions modulated by (almost) arithmetically global data associated to the original representation. I will describe how this works at height 1 in a moment; generalizing this to height 2 and up is the first major goal of this program.

Notice that, in order to completely capture the behavior of these representations, we need to incorporate all level structures (due to ramification issues). A similar phenomenon is expected to occur in the chromatic case, which necessitates the inclusion of *modular equivariance*. That is to say, we would like to define for each n some notion of "TAF_n with level structure", where the various levels assemble into some kind of genuine equivariant E_∞-ring. This is done using "Drinfeld level structures", which generalize the notion of a level structure as a certain kind of discrete subgroup.

At height 1, this is almost classical: modular-equivariant K-theory is Atiyah's "real K-theory", the C_2-equivariant ultracommutative ring spectrum denoted $K\mathbb{R}$. This spectrum arises as the global sections of the TAF sheaf on $B\mathbb{G}_m$, with global sections given by KO and sections on the double cover given by KU. Since the action is given on $\mathbb{G}_m$ by multiplication by -1, which is invertible, this can be enhanced to a sheaf with transfers and thus yields an equivariant ring. This theory admits an arithmetic interpretation in terms of the Adams conjecture and the Riemann zeta function; see [1] and [15].

The height 2 analogue of this is, however, more problematic. Over the integers, the stack $\widehat{\mathcal{M}}_{ell}$ has fundamental group C_2, with the only nontrivial connected cover given by inversion on elliptic curves. This is because a level $N > 0$ structure on an elliptic curve roughly corresponds to an isogeny of index N, which will be ramified (at every point) unless N is invertible. The more primes we invert, the larger the fundamental group will get, all the way through the classical complex version with fundamental group $SL_2(\mathbb{Z})$. So to get every level structure without destroying the requisite power operations, we need to rationalize. But that destroys all the information we're trying to get!

We have a couple of options here. We can remove the isogenies of bad index, as Davies does in [8], but that still loses some information. We can invert primes "situationally" using the method of [14], which loses as little information as possible while still constructing this as an "ordinary" equivariant spectral stack. Or, last but not least, we can follow Atiyah's approach to Adams operations and try to incorporate global equivariance. This is probably the right answer, but it will require some new definitions for equivariant spectral stacks and some machinery to "combine" the equivariances. This is likely to involve a certain known map from the Strickland ring St_n, which classifies Σ_n-equivariance, to the Drinfeld ring, D_A, which classifies modular equivariance. The details, however, are not yet known.

2. Conformal Field Theory

An n-dimensional QFT is a symmetric monoidal functor $Z : \mathrm{Cob}_{n-1,n} \to \mathrm{Vect}$ (or Hilb). We can replace our cobordism category with $\mathrm{Cob}_{(d,d+n}$ $(1 \leq n \leq \infty)$, the symmetric monoidal (∞, n)-category of cobordisms between d-manifolds, in which case we have to replace Vect with some appropriate "n-category of vector spaces" as well. It isn't known how to do this in general, but at least for the case $n = 2$ we can replace it with the Morita 2-category of algebras.

We can choose different cobordism categories by equipping our manifolds with different geometric structure. If we take oriented smooth manifolds, we get TQFTs. If we take oriented manifolds with conformal structure, we get CFTs. Often, we can do this for a large class of manifolds in a compatible manner using a particular construction called a "quantum σ-model", which is generally constructed by some kind of geometric procedure that works for various manifolds[3]. In this case, we can take the symmetric monoidal trace of the theory, which is the value of Z on a $d+1$-dimensional torus. (Physically, this can be interpreted as a probability amplitude for the creation and annihilation of a pair of d-branes.) This is the topologist's version of the "partition function", and gives rise to a genus, i.e. a ring homomorphism from the classical cobordism ring Ω (for some notion of cobordism) to a specified target ring. Cases of particular interest include the $\hat{A}$-genus for spin manifolds (which is the "height 1" version and describes a spinning particle) and the Witten genus for string manifolds (c.f. Ochanine) (which is the "height 2" version and describes a superstring). Both of these cases arise from the quantization of σ-models; see [16].

These cases have the nice property that they lift to orientations of ring spectra, specifically the ring spectra KO and tmf. In fact, this is the origin of tmf: Witten conjectured its existence as a spectral lift of modular forms that would serve as a target for a spectral Witten genus. The spin orientation of K-theory was originally constructed by Atiyah-Bott-Shapiro using Clifford algebras ([3]), later shown to be E-infinity by Joachim in 2004 ([10]). The string orientation of tmf was constructed (as an E-infinity orientation from the start) by Ando-Hopkins-Rezk in 2010 ([2]).

Computational evidence suggests that these genera are related. If we assume a notion of S^1-equivariant spin structure and formally apply the localiation theorem for equivariant K-theory, we find that a string structure on M is equivalent to an equivariant spin structure on its free loopspace $\mathcal{L}M$ (which we think of as an "infinitesimal thickening" of M; although that should really be the *formal* loop space, but I'm not opening that can of worms). To be precise, what Witten argues is that that the equivariant index of the Dirac operator on $\mathcal{L}M$ is the Ochanine genus of M. This is not quite rigorous, so it invites attempts at formalization. The fact that the targets are both a form of TAF (and we can also get versions valued in e.g. tmf with level structure as well as a version at height 0) suggests that we should think of this as some kind of arithmetically global transchromatic phenomenon.

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[3]Though they are often constructed in a uniform manner, σ-models make perfect sense if we fix a manifold.

3. Transchromatic Ramification

The algebraic K-theory of an E_∞-ring R, $K(R)$, is defined as the group completion of the core of its compact module category. Note that we can extend this to spectral stacks by taking the category of coherent sheaves. (This is traditionally done using algebraic vector bundles, but taking coherent sheaves is equivalent for connective rings, so we'll use that definition here.) Note that the sphere spectrum is the algebraic K-theory of the "field with one element" in the sense that it is the K-theory of the category of finite sets. This is another hint towards the relationship between stable homotopy theory and number theory discussed in section 1. The main reason algebraic K-theory is relevant to the chromatic story, however, is the following "Redshift theorem".

Theorem 3.1. *If R is an E_∞-ring of height $n \geq 0$, $K(R)$ has height $n+1$.*

This theorem was proven last year in [7].

Algebraic K-theory is very hard to compute, so we use "trace methods" to approximate it.

Definition 3.2. Let R be an E_∞-ring. The *topological Hochschild homology* of R, $THH(R)$, is the geometric realization of the simplicial E_∞-ring given by the bar complex of R.

We are interested in an S^1-equivariant version of this. We definte a *cyclotomic spectrum* to be a spectrum with S^1-action equipped with S^1-equivariant isomorphisms $E \simeq E^{tC_p}$ for all primes p. Here E^{tC_p} is the Tate fixed-point spectrum E^{hC_p}/E_{hC_p}, and the isomorphism is called the *Frobenius*. (We can equivalently define a cyclotomic spectrum to be a genuine S^1-spectrum, in which case we replace the Tate fixed-points with the ordinary homotopy fixed-points.)

Lemma 3.3. *$THH(R)$ has a natural cyclotomic structure induced by permutations of the bar complex.*

Definition 3.4. The *topological cyclic homology* of R is the spectrum $TC(R) = \mathrm{Map}(\mathbb{S}^{\mathrm{triv}}, THH(R))$, where the mapping spectrum is taken in the category of cyclotomic spectra.

There is a natural map $K \to TC$ called the *cyclotomic trace*. There is also a natural "forgetful" map $TC \to THH$, and the composition of these two is a map $K \to THH$ called the *Dennis trace*. Ayala-Mazel-Gee-Rozenblyum give a geometric interpretation of this ([4]), where these notions are all extended to stacks in the evident way.

Theorem 3.5. *Let X be a spectral stack. Then we have a natural identification $THH(X) \simeq \mathcal{O}(\mathcal{L}X)$. This identification induces an identification of $TC(X)$ with, roughly speaking, the S^1-equivariant functions on $\mathcal{L}X$ compatible with all iterates $S^1 \to S^1 \to X$ up to "universal indeterminacy".*

Basically, TC is like THH, but equivariant with respect to all transformations of S^1, including rotations, folds, etc. Note that this suggests we should think of TC as the global sections of the equivariant free loop space of a ring—precisely the space that appears in Witten's equivariant index conjecture!

They also give us a nice geometric interpretation of the trace in terms of monodromy. I won't state the theorem here, but I'll describe the idea. Let E be a vector bundle, and choose a loop γ, say based at a point x. Then γ induces a monodromy automorphism $E_x \to E_x$. If we take the trace of this automorphism (in the basic linear algebra sense), we have defined a function on $\mathcal{L}X$ which is in some sense cyclotomically invariant. To be specific, if we take a collection of loops $\gamma_1, \dots, \gamma_n$ whose composition is γ, the trace will be unchanged by cyclic permutation of these loops. This follows from the first thing you ever learned about trace in linear algebra, namely its cyclic invariance, and says that our function respects the natural operations on loops. *This is the cyclotomic trace.*

Traditionally, trace can be used in algebraic number theory to detect separability and therefore ramification.

Theorem 3.6. *A finite field extension L/K is separable iff $\mathrm{Tr}_{L/K}$ is not identically zero.*

More generally, we can detect ramification using Hochschild homology.

Theorem 3.7. *An algebra A/k is separable iff $HH^k_*(A, M) = 0$ for any bimodule M. (Here HH^k denotes Hochschild homology over k.)*

Remark 3.8. The triviality of Hochschild homology here is equivalent to the statement that the Hochschild homologies of k and A are equivalent, which generalizes the nondegeneracy condition for the trace.

This theory also exists in a spectral context due to Berman ([6]).

Definition 3.9 (Berman)**.** Let $* \to Y \to X$ be maps of spectral schemes (where $*$ is any scheme but can be thought of as a point). Write f for the induced map $\Omega_* Y \to \Omega_* X$ (where the loop spaces are defined as pullbacks in the evident way), and i for the induced map $* \to \Omega_* X$. Take $\mathcal{F}$ to be the fiber of $\mathcal{O}(\Omega_* X) \to f_* \mathcal{O}(\Omega_* Y)$, a sheaf on $\Omega_* X$. That is, $\mathcal{F}$ is the sheaf of "functions on $\Omega_* X$ which vanish on loops that lift to Y". Then we say Y/X is

(1) *Unramified* if $i^* \mathcal{F} \simeq 0$ (i.e. $\mathcal{F}$ is trivial at constant loops)

(2) *Totally ramified* if $\mathcal{F} \simeq i_*(\mathcal{F}_0)$ for some sheaf $\mathcal{F}_0$ on $*$ (i.e. $\mathcal{F}$ is trivial away from constant loops).

We should think of the loops here as analogous to the residue class field of a local field, both being a home for infinitesimal extensions. Then this appears directly analogous, since a finite extension of local fields is totally ramified iff the extension of residue class fields is trivial, and unramified if the residue degree is equal to the degree of the extension. In fact, this is more than just an analogy.

Theorem 3.10 (Berman)**.** *When restricted to number fields, this coincides with the classical definition of unramified and totally ramified extensions.*

Berman tells us that if an extension splits into unramified and totally ramified parts, we can compute the ramified part by "ramified descent".

Theorem 3.11 (Berman)**.** *Write $Ram^S(A/R)$ for the fiber of $THH^S(R) \otimes_R A \to THH^S(A)$ (where the superscript denotes relative Hochschild homology of an extension $R \to A$ over the base ring S), and let $A \to k$ be a k-point of* $\operatorname{Spec} A$. *Write $I^k_{A/R}$ for the fiber of the extension map $k \otimes_R k \to k \otimes_A k$. Then A/R is unramified at k iff $Ram^R(A/R) \otimes_A k = 0$; and if it is totally ramified at k, then $Ram^S(A/R) \otimes_A k = THH^S(k) \otimes I^k_{A/R}$.*

The geometric interpretation of K/TC/THH suggests that this notion of ramification should lift to TC and the cyclotomic trace. This has applications to transchromatic homotopy, since increasing chromatic height can be thought of as splitting into totally ramified and unramified parts via the connected-etale sequence. Also, we usually present the universal deformation about the height-n stratum (at a fixed prime) as the ring Landweber-classified by the Lubin-Tate formal group, which classifies ramified abelian extensions of local fields. So we think that applying K to an arithmetically global height n cohomology theory should give us an arithmetically global height $n+1$ cohomology theory which splits into the height exactly $n+1$ part and height $\leq n$ part, of which the first is ramified and the second unramified. This is like how Tate K-theory is an elliptic cohomology theory, but also a straightfoward extension of KO; and, moreover, it is classified by a degenerate elliptic curve, so it makes sense that it has a "singular" relationship to tmf. This is witnessed by the splitting of Hecke operators for modular forms into a sum of the Atkin operators (which is a decategorification of the Adams operations) and an operator that rotates $\widehat{\mathcal{M}_{ell}}$ about ∞, the point classifying the degenerate elliptic curve.

But, since TC classifies monodromy on the cyclotomic loop space, this splitting should be something that can be described using the same formalism as Witten's equivariant index theory! In fact, Witten himself has phrased this in terms of ramification, and used similar ideas (minus the higher algebra and chromatic homotopy) to realize geometric Langlands strictly in terms of quantum field theory. Lifting this perspective to higher algebra, therefore, is a highly promising endeavor.

References

[1] J. F. Adams. "On the Groups J(X)—IV". In: *Topology* 5.1 (Mar. 1, 1966), pp. 21–71. ISSN: 0040-9383. DOI: 10.1016/0040-9383(66)90004-8. URL: https://www.sciencedirect.com/science/article/pii/0040938366900048 (visited on 03/28/2023).

[2] Matthew Ando, Michael J Hopkins, and Charles Rezk. *MULTIPLICATIVE ORIENTATIONS OF KO-THEORY AND OF THE SPECTRUM OF TOPOLOGICAL MODULAR FORMS*. preprint.

[3] M. F. Atiyah, R. Bott, and A. Shapiro. "Clifford Modules". In: *Topology* 3 (July 1, 1964), pp. 3–38. ISSN: 0040-9383. DOI: 10.1016/0040-9383(64)90003-5. URL: https://www.sciencedirect.com/science/article/pii/0040938364900035 (visited on 05/19/2022).

[4] David Ayala, Aaron Mazel-Gee, and Nick Rozenblyum. *The Geometry of the Cyclotomic Trace*. Oct. 17, 2017. DOI: 10.48550/arXiv.1710.06409. arXiv: arXiv:1710.06409. URL: http://arxiv.org/abs/1710.06409 (visited on 02/28/2023). preprint.

[5] Mark Behrens and Tyler Lawson. *Topological Automorphic Forms*. Dec. 11, 2008. arXiv: arXiv:math/0702719. URL: http://arxiv.org/abs/math/0702719 (visited on 09/01/2022). preprint.

[6] John D. Berman. *Ramification and Descent in Homotopy Theory and Derived Algebraic Geometry*. Dec. 29, 2021. DOI: 10.48550/arXiv.2112.14568. arXiv: arXiv:2112.14568. URL: http://arxiv.org/abs/2112.14568 (visited on 02/28/2023). preprint.

[7] Robert Burklund, Tomer M. Schlank, and Allen Yuan. *The Chromatic Nullstellensatz*. July 20, 2022. DOI: 10.48550/arXiv.2207.09929. arXiv: arXiv:2207.09929. URL: http://arxiv.org/abs/2207.09929 (visited on 08/12/2022). preprint.

[8] Jack Morgan Davies. *Hecke Operators on Topological Modular Forms*. Dec. 12, 2022. DOI: 10.48550/arXiv.2212.06208. arXiv: arXiv:2212.06208. URL: http://arxiv.org/abs/2212.06208 (visited on 12/15/2022). preprint.

[9] Michael Hill and Tyler Lawson. *Topological Modular Forms with Level Structure*. Feb. 3, 2015. DOI: 10.48550/arXiv.1312.7394. arXiv: arXiv:1312.7394. URL: http://arxiv.org/abs/1312.7394 (visited on 03/13/2023). preprint.

[10] Michael Joachim. "Higher Coherences for Equivariant K-theory". In: *Structured Ring Spectra*. Ed. by Andrew Baker and Birgit Richter. London Mathematical Society Lecture Note Series. Cambridge: Cambridge University Press, 2004, pp. 87–114. ISBN: 978-0-521-60305-8. DOI: 10.1017/CBO9780511529955.006. URL: https://www.cambridge.org/core/books/structured-ring-spectra/higher-coherences-for-equivariant-ktheory/FFFD8CB4BDF7BBB54E6079BB3EA5CC18 (visited on 03/10/2023).

[11] Tyler Lawson and Niko Naumann. *Strictly Commutative Realizations of Diagrams over the Steenrod Algebra and Topological Modular Forms at the Prime 2*. Jan. 15, 2013. DOI: 10.48550/arXiv.1203.1696. arXiv: arXiv:1203.1696. URL: http://arxiv.org/abs/1203.1696 (visited on 03/13/2023). preprint.

[12] Aaron Mazel-Gee. "Goerss–Hopkins Obstruction Theory via Model ∞-Categories". UC Berkeley. 545 pp. URL: https://etale.site/writing/GHOsT-thesis.pdf (visited on 03/13/2023).

[13] John Rognes. *Galois Extensions of Structured Ring Spectra*. Dec. 6, 2005. DOI: 10.48550/arXiv.math/0502183. arXiv: arXiv:math/0502183. URL: http://arxiv.org/abs/math/0502183 (visited on 08/12/2022). preprint.

[14] A. Ronan. *Nilpotent G-spaces and Their Localizations*. Jan. 23, 2023. DOI: 10.48550/arXiv.2301.09691. arXiv: arXiv:2301.09691. URL: http://arxiv.org/abs/2301.09691 (visited on 01/25/2023). preprint.

[15] A. Salch. *Denominators of Special Values of Zeta-Functions Count KU-local Homotopy Groups of Mod p Moore Spectra*. Mar. 16, 2023. DOI: 10.48550/arXiv.2303.09550. arXiv: arXiv:2303.09550. URL: http://arxiv.org/abs/2303.09550 (visited on 03/28/2023). preprint.

[16] Edward Witten. "The Index of the Dirac Operator in Loop Space". In: *Elliptic Curves and Modular Forms in Algebraic Topology*. Springer, 1988, pp. 161–181.